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ON THE DEFINITION OF WORD HYPERBOLIC GROUPS

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ABSTRACT. Formal languages based on multiplication tables of finitely generated groups are investigated and used to give a linguistic characterization of word hyperbolic groups.

1. Introduction

Over the last several years combinatorial group theory has been influenced by ideas from both low dimensional topology and formal language theory. Applications of language theory include the classification of groups with context–free word problem [15] (together with [5]), the use of indexed languages to describe the fundamental groups of the known compact 3–manifolds [3], a forthcoming complexity–theoretic analog of the Higman embedding theorem [2], and the general theory of automatic groups [6]. In this paper we use formal languages in a novel way to obtain a linguistic characterization of word hyperbolic groups.

A formal language is a subset of a free monoid Σ^* over a finite alphabet Σ. The connection between between a group G and languages over Σ is made by means of a surjective monoid homomorphism $\Sigma^* \to G$ which maps $w \in \Sigma^*$ to $\overline{w} \in G$. The usual languages considered are the word problem, $\{w \mid \overline{w} = 1\}$, and combings, i.e., languages projecting onto G. We consider instead languages derived from the multiplication table of G. For this purpose we need a new letter # not in the alphabet Σ.

Theorem 1. *Let $\Sigma^* \to G$ be a choice of generators for the group G. G is word–hyperbolic if and only if for some regular combing $R \subset \Sigma^*$, the language $M = \{u\#v\#w \mid u, v, w \in R, \overline{uvw} = 1\}$ is context–free.*

In short G is hyperbolic if and only if it has a context–free multiplication table. It is interesting that the original geometric definition of word hyperbolic groups in terms of the thin triangle condition is equivalent to a purely language-theoretic definition. Choices of generators are defined in Section 2.

For any combing R call $M = \{u\#v\#w \mid u, v, w \in R, \overline{uvw} = 1\}$ the multiplication table determined by R. We investigate multiplication tables of virtually free and automatic groups. Theorem 2 is a variation on the main result of [15].

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Theorem 2. *Let $\Sigma^* \to G$ be a choice of generators and M the multiplication table corresponding to the combing $R = \Sigma^*$.*

1. *G is finite if and only if M is a regular language.*
2. *G is virtually free if and only if M is context–free.*

In Theorem 3 we consider columns of the multiplication table. The column of $g \in G$ is $C(g) = \{u\#w \mid u, w \in R, \overline{u}g\overline{w} = 1\}$. Columns are related to the comparator automata used in the definition of automatic groups. Suppose G is automatic with respect to the combing used to define M, and $a \in \Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ where ϵ is the empty word. The binary relation accepted by the comparator automaton for a is $\{(u, w) \mid u, w \in R, \overline{ua} = \overline{w}\}$ while $C(\overline{a}) = \{u\#w \mid u, w \in R, \overline{uaw} = 1\}$.

Theorem 3. *Let $\Sigma^* \to G$ be a choice of generators. There exists a combing $R \subset \Sigma^*$ such that $C(\overline{a})$ is context–free for all $a \in \Sigma_\epsilon$ if and only if G is asynchronously automatic with respect to a combing contained in Σ^* and closed under taking formal inverses.*

Groups which are asynchronously automatic with respect to a combing closed under formal inverses form a subclass of asynchronously biautomatic groups. It does not seem to be known whether or not this subclass is proper.

The proof of Theorem 1 depends on the fact that the thin triangle condition can be relaxed. The distance from a point on one side of a triangle to the union of the other two sides may be allowed to grow with the size of the triangle, and the sides of the triangle need not be geodesics. See Theorem 8 in Section 3.

Another linguistic characterization of hyperbolic groups is given by Grunschlag [12, Section 3.2]. He shows that hyperbolic groups are those whose word problem is generated by a terminating growing context–sensitive grammar.

Hyperbolic groups were introduced by Gromov [11]. Additional references are [1], [4] and [8].

2. Preliminary Items

Keep the notation introduced in Section 1. G is a finitely generated group, Σ is a finite alphabet, and # is a letter not in Σ. $\Sigma_\# = \Sigma \cup \{\#\}$, and $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ where ϵ stands for the empty word. Σ has formal inverses if it admits a permutation $a \to a^{-1}$ with orbits of length two. Formal inverses on Σ extend to formal inverses on Σ^* by means of the rule $(wv)^{-1} = v^{-1}w^{-1}$.

2.1. **Choice of Generators.** A choice of generators for G consists of a finite alphabet Σ equipped with formal inverses together with a surjective monoid homomorphism $\Sigma^* \to G$ which maps w^{-1} to $\overline{w}^{-1}$. Any choice of generators $\Sigma^* \to G$ is extended to $\Sigma_\#^* \to G$ via $\overline{\#} = 1$.

Given a choice of generators $\Sigma^* \to G$ we define a path of length n in G to be a sequence $g_0, \ldots, g_n$ of of group elements such that $g_i = g_{i-1}\overline{a}_i$ for some $a_i \in \Sigma$. The label of this path is $a_1 \cdots a_n \in \Sigma^*$. We use the usual arrow

notation $\cdots g_{i-1} \xrightarrow{a_i} g_i \cdots$. On occasion we will allow paths with labels in $\Sigma_{\#}^*$.

Each word in $\Sigma_{\#}^*$ determines a path up to left–translation by G. We identify words with paths and specify a particular path corresponding to a word when necessary. The length of the shortest path from g to h is $d(g, h)$, a left–invariant metric on G. Shortest paths are called geodesics.

2.2. **Triangles.** A triangle T consists of three points in G joined by paths with labels in Σ^*. These paths are the sides of T. T is δ–thin if the distance from any point on one side to the union of the other two sides is at most δ. The width of T, $\delta(T)$, is the smallest number δ for which T is δ–thin. The norm of T, $|T|$, is the maximum distance between its vertices. For any language $L \subset \Sigma^*$, T is an L-triangle if its sides are in L. In particular T is a geodesic triangle if its sides are geodesics. If $\delta(T)$ is bounded as T ranges over geodesic triangles, then G satisfies the thin triangle condition and is word hyperbolic.

Each element $u\#v\#w$ of the multiplication table corresponding to a combing R determines up to translation by G an R–triangle with sides u, v, w. For brevity we may refer to $u\#v\#w$ itself as a triangle.

2.3. **Formal Languages.** See [13],[14],[16],[17] for standard introductions to the theory of automata and formal languages; a group theoretic perspective is available in [10].

Recall that context–free languages are the languages generated by context–free grammars and that a context–free grammar $\mathcal{G}$ consists of a terminal alphabet Σ (with or without formal inverses), a set of nonterminals N, a start symbol $S \in N$, and a set of productions of the form $A \to \alpha$ where $A \in N$ and $\alpha \in (\Sigma \cup N)^*$. All these sets are finite.

Elements of $(\Sigma \cup N)^*$ are called sentential forms. The notation $\alpha \to \beta$ means that the lefthand side of some production is a subword of the sentential form α, and that the sentential form β is obtained by replacing that subword by the righthand side of the production. The effect of zero or more replacements is denoted by $\alpha \xrightarrow{*} \beta$. When $\alpha \xrightarrow{*} \beta$ we say β is derived from α or α derives β. The language generated by $\mathcal{G}$ is $\{w \mid w \in \Sigma^*, S \xrightarrow{*} w\}$.

2.4. **Transductions and Rational Subsets.** A rational transduction $\rho : \Sigma^* \to \Delta^*$ from one finitely generated free monoid to another is a rational subset of $\Sigma^* \times \Delta^*$. Write $\rho(w) = v$ if $(w, v) \in \rho$ and $\rho(L) = \{v \mid \exists w \in \Sigma^* \rho(w) = v\}$ for $L \subset \Sigma^*$. The inverse of ρ is $\rho^{-1} = \{(v, w) \mid (w, v) \in \rho\}$.

The rational subsets of any monoid P are the closure of its finite subsets under union, product, and generation of submonoid. Equivalently rational subsets are the subsets accepted by finite automata over P. A finite automaton $\mathcal{A}$ over P is a finite directed graph with edge labels from P, a distinguished initial state, and some distinguished terminal states. $\mathcal{A}$ accepts the set of labels of paths which begin at the initial state and end at a terminal state. Automata may be allowed to have more than one initial

state. The accepted set is a union of sets accepted by automata with unique initial states and so is rational.

Rational subsets of Σ^* are called regular languages. It is customary to restrict edge labels in automata over Σ^* to Σ or Σ_ϵ, but this restriction is not necessary. Regular languages are closed under intersection and difference while rational sets in general are not.

Since rational transductions are rational subsets, they are closed under union, product and generation of submonoids. They are also closed under inverse and under composition in the sense of binary relations. Images of regular and context–free languages under rational transductions are regular and context–free respectively. In particular regular and context–free languages are closed under homomorphism, inverse homomorphism, and intersection with regular languages. Rational transductions are not closed under intersection, but if $\rho : \Sigma^* \to \Delta^*$ is a rational transduction, $R \subset \Sigma^*$ is regular, and $S \subset \Delta^*$ is also regular, then $\rho \cap (R \times S)$ is a rational transduction.

Lemma 4. *Fix $w, v \in \Sigma^*$; $\rho = \{(xwy, xvy) \mid x, y \in \Sigma^*\}$ is a rational transduction from Σ^* to itself.*

Proof. Let D be the diagonal submonoid of $\Sigma^* \times \Sigma^*$. Since D is finitely generated, it is rational. It follows that $\rho = D(w, v)D$ is a product of rational sets and so is itself rational. □

Lemma 5. *If Σ and Δ have formal inverses and $\rho : \Sigma^* \to \Delta^*$ is a rational transduction, then so is $\tau = \{(w, v) \mid (w^{-1}, v^{-1}) \in \rho\}$.*

Proof. Pick an automaton accepting ρ. Reverse the orientation of each edge and invert the edge label. Make every terminal state an initial state and every initial state a terminal state. □

Lemma 6. *Let Σ have formal inverses. A relation $\rho : \Sigma^* \to \Sigma^*$ is a rational transduction if and only if $L = \{u\#w \mid (u, w^{-1}) \in \rho\}$ is generated by a context–free grammar with all productions of the form $A \to xBy$ or $A \to x\#y$ for $x, y \in \Sigma^*$.*

Proof. Suppose ρ is accepted by a finite automaton $\mathcal{A}$ over $\Sigma^* \times \Sigma^*$. Construct a context–free grammar with one nonterminal A_p for each vertex p of $\mathcal{A}$. The start symbol is the nonterminal corresponding to the initial vertex. For each edge $p \xrightarrow{(x,y)} q$ there is a production $A_p \to xA_q y^{-1}$, and for each terminal vertex q there is another production $A_q \to \#$. It is straightforward to check that this grammar generates L. The main step is to use induction on path length and on derivation length to prove that $A_p \xrightarrow{*} uA_q w$ if and only if there is a path in $\mathcal{A}$ from p to q with label (u, w^{-1}).

For the converse suppose L is generated by a context–free grammar $\mathcal{G}$ as above. Construct an automaton $\mathcal{A}$ whose vertices are the nonterminals of $\mathcal{G}$ plus one terminal vertex. The initial vertex is the start symbol. For each production $A \to xBy$ there is an edge $A \xrightarrow{(x,y^{-1})} B$, and for each production

$A \to x\#y$ there is an edge with label (x, y^{-1}) from A to the terminal vertex. Again it is straightforward to check that $u\#w \in L$ if and only if $\mathcal{A}$ accepts (u, w^{-1}). □

Lemma 7. *G is asynchronously automatic with respect to a regular combing R if and only if for all $a \in \Sigma_\epsilon$ the relation $\rho_a = \{(u, w) \mid u, w \in R, \overline{ua} = \overline{w}\}$ is a rational transduction.*

Proof. If G is asynchronously automatic, then by [6, Definition 7.2.1] ρ_a is a rational transduction. The converse is [18, Theorem 1] except that the automata used there are more restricted than ours. In terms of our notation the vertices of those automata are partitioned into two sets. All edges leaving the first set have labels from $\Sigma_\epsilon \times \{\epsilon\}$, and all edges leaving the second set have labels from $\{\epsilon\} \times \Sigma_\epsilon$.

An automaton in our sense can be transformed into one satisfying the definition in [18]. Replace edges by paths if necessary to insure that edge labels are from $(\Sigma_\epsilon \times \{\epsilon\}) \cup (\{\epsilon\} \times \Sigma_\epsilon)$. If vertex p is a source for edges of both types, add a vertex p' and make all the edges of one type start at p' instead of p. Add edges from p to p' and p' to p with label (ϵ, ϵ).

There is one more detail. In [18] ρ_a is defined as $\{(u\$, w\$) \mid u, w \in R, \overline{u}a = \overline{w}\}$ instead of $\{(u, w) \mid u, w \in R, \overline{u}a = \overline{w}\}$, but if one version of ρ_a is a rational transduction, then the other one is too. □

3. Flabby Triangles

In preparation for the proof of Theorem 1 we show that the thin triangle condition used to define word hyperbolic groups can be weakened.

Theorem 8. *A group G is word–hyperbolic if it admits a choice of generators $\Sigma^* \to G$ and a combing $R \subset \Sigma^*$ such for some constant C every R–triangle T in G has width $\delta(T) \leq |T|/75 + C$.*

The rest of this section is devoted to proving Theorem 8. Without loss of generality assume that there is just one combing path for each $g \in G$. By [9, Theorem B] it suffices to show that for some constant K every cycle in G of length n can be triangulated with diagonals of length at most $n/6 + K$. Before discussing triangulations we prove a lemma modeled on [4, Lemma 1.5 of Chapter 3].

Lemma 9. *Let w and v_0 be paths in G from g to h with $|w| = n$ and $v_0 \in R$. There is a constant D independent of n such that every point on v_0 is a distance at most $n/36 + D$ from w. If w is a geodesic, then every point of w is a distance at most $n/18 + 2D$ from v_0.*

Proof. Let p_0 be a point on v_0. If $n \leq 2$, then the distance from p_0 to w is at most E, the maximum length of the finitely many combing paths for elements $g \in G$ with $d(1, g) \leq 2$. Otherwise estimate the distance by constructing a sequence of triangles as in Figure 1. Let T_0 be an R–triangle whose base is v_0 and whose third vertex is a point as close to the middle

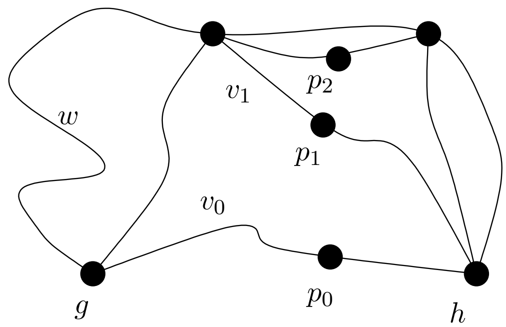

FIGURE 1. A sequence of R–triangles.

of w as possible. One side, call it v_1, of T_0 is distinct from v_0 and contains a point p_1 with $d(p_1, p_0) \leq \delta(T_0)$. If v_1 subtends a segment of w of length greater than 2, construct triangle T_1 with base v_1 in the same way T_0 was constructed. Continue until reaching a triangle T_m with point p_{m+1} on a side subtending a segment of w of length at most 2.

To show that the sequence of triangles terminates consider the sequence of numbers defined by $b_0 = n$ and $b_{k+1} = (1 + b_k)/2$. From the construction above it is clear that the base of triangle T_k subtends a segment of w of length at most b_k. It is straightforward to show that $b_k \leq 1 + n/2^k$ whence the sequence of triangles stops at T_m for some $m \leq \log_2(n)$.

Since the third vertex of each T_k lies on the segment of w subtended by the base of T_k, we have $|T_k| \leq b_k$. The distance from p_0 to w is at most $E + \sum \delta(T_k) \leq E + \sum(b_k/75 + C) \leq E + 2n/75 + (1+C)(1+\log_2 n) \leq n/36 + D$ for some constant D.

To verify the last assertion of the lemma assume that w is a geodesic and set $A = n/36 + D$. The points on w a distance greater than A from v_0 form a union of subpaths of w not containing g or h. Let u be any such subpath, and write $w = w_1 u w_2$. Observe that w_1 starts at g, w_2 ends at h, and each point of v_0 is a distance at most A from w_1 or w_2. It follows that there are two adjacent points g_1, g_2 on v_0 and points h_i on the paths w_i such that $d(g_i, h_i) \leq A$. Since w is a geodesic, the distance along w from h_1 to h_2 is $d(h_1, h_2) \leq d(h_1, g_1) + 1 + d(g_2, h_2) \leq 2A + 1$. But then any point on u is a distance at most $2A$ from v_0. $\square$

We continue with the proof of Theorem 8. Recall that it suffices to triangulate w, a cycle of length n in Γ, with diagonals of length at most $n/6 + K$. Take $K = 6D + 3$ where D is the constant from Lemma 9.

To triangulate w realize it as a regular polygon P in the plane. The vertices of P are labelled by the group elements $g_1, \ldots, g_n$ which occur along w, and the edges are labelled by the letters of w. Particular group elements may occur more than once as labels. If $n \leq 3$, then w is deemed to be triangulated without any diagonals. Otherwise join the vertices of P in pairs by diagonals, i.e., straight line segments, so that no two diagonals

meet in the interior of P; the interior is divided into triangles; and each edge of P is one side of a triangle. The length of a diagonal is the distance in G between the labels of its endpoints. Edge lengths are defined similarly and are either 0 or 1.

If $|w| > 3$, w can be triangulated in the following way so that all diagonals have length at most $n/6 + K$. First add a diagonal from g_n to g_2; this diagonal has length at most 2. We are done if $n = 4$. Otherwise it suffices to show that whenever a diagonal of length at most $n/6 + K$ has endpoints g_i, g_j with $3 \le j - i$, then we can add a diagonal from g_i to g_{j-1} or one from g_{i+1} to g_j or diagonals from g_i and g_j to some g_k with $i + 2 \le k \le j - 2$. In other words it suffices that $d(g_i, g_k)$ and $d(g_j, g_k)$ are at both most $n/6 + K$ for some k with $i < k < j$.

Pick $h \in G$ as close as possible to the middle of a geodesic path from g_i to g_j. By Lemma 9 h is a distance at most $n/18 + 2D$ from some point on the R–path from g_i to g_j, and that point is itself a distance at most $n/36 + D$ from the segment w' of w beginning at g_i and ending at g_j. Consequently h is a distance at most $n/12 + 3D$ from w', and it follows that $d(h, g_k) \le n/12 + 3D + 1$ for some g_k with $i < k < j$. Hence $d(g_i, g_k) \le d(g_i, h) + d(h, g_k) \le (1/2)(n/6 + K + 1) + n/12 + 3D + 1 \le n/6 + K$, and likewise for $d(g_j, g_k)$.

4. Hyperbolic Implies Context–Free

With respect to any choice of generators $\Sigma \to G$ the geodesic combing R of a hyperbolic group G is a regular language [8, Theorem 13 in Chapter 9]. In this section we show that the multiplication table M determined by R is context–free.

Words $r\#s\#t \in M$ correspond to geodesic triangles whose sides are paths with labels r, s, t. By [8, Proposition 21 in Chapter 2] we may choose $\delta \ge 1$ so that points on the perimeter of each geodesic triangle match in pairs with each point corresponding to another point an equal distance from one of the vertices and matching points a distance at most δ apart. Figure 2 shows a geodesic triangle with edge labels $a_1 \cdots a_p$, The interior arrows indicate paths in G between matching points on the sides. The fact that points along the perimeter match in pairs implies $i + j = p$, $j + k = q$, and $i + k = r$. $b_1 \cdots b_q$, $c_1 \cdots c_r$.

Define a context–free grammar $\mathcal{G}$ whose terminal alphabet is $\Sigma_\#$ and whose nonterminal alphabet N consists of a symbol X_w for each word $w \in \Sigma_\#^*$ of length $|w| \le \delta$. Let $V = \Sigma_\# \cup N$ and extend the choice of generators to a monoid homomorphism $V^* \to G$ by $X_w \to \overline{X}_w = \overline{w}$ and $\overline{\#} = 1$. The start symbol of $\mathcal{G}$ is X_ϵ.

The productions of $\mathcal{G}$ are all replacements $X \to \alpha$ with $X \in V$, α a word of length at most 5 in V, and $\overline{X} = \overline{\alpha}$. Since applying productions does not change images in G, it is clear that $\mathcal{G}$ generates a context–free language of words defining the identity in G.

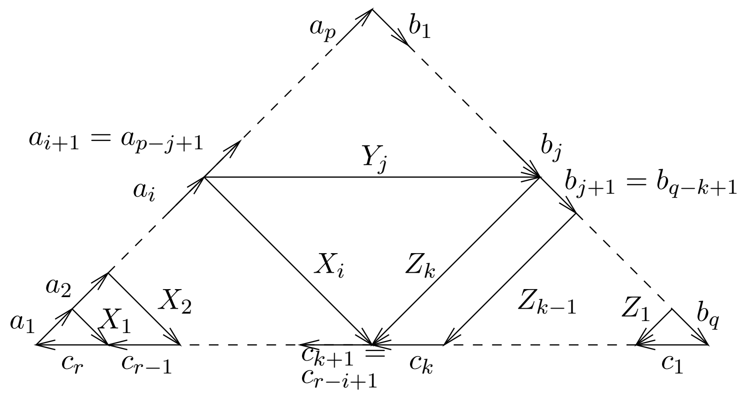

FIGURE 2. A geodesic triangle.

To obtain a leftmost derivation of $a_1 \cdots a_p \# b_1 \cdots b_q \# c_1 \cdots c_r$ from Figure 2 begin with $X_\epsilon \to a_1 X_1 c_r$ and continue with productions corresponding to inscribed quadrilaterals.

$$X_\epsilon \to a_1 X_1 c_r \to a_1 a_2 X_2 c_{r-1} c_r \xrightarrow{*} a_1 \cdots a_i X_i c_{r-i+1} \cdots c_r$$

Apply the production $X_i \to Y_j Z_k$, and then do

$$Y_j \to a_{i+1} Y_{j-1} b_j \xrightarrow{*} a_{i+1} \cdots a_{p-1} Y_1 b_2 \cdots b_{q-k} \to a_{i+1} \cdots a_p \# b_1 \cdots b_j.$$

Treat Z_k similarly.

In the preceding derivation the right–hand sides of all productions have length at 3. The reason we require productions with longer right–hand sides is that in some geodesic triangles the central figure is a hexagon instead of a triangle. For that case we need productions $X \to aYbZc$ as in Figure 3.

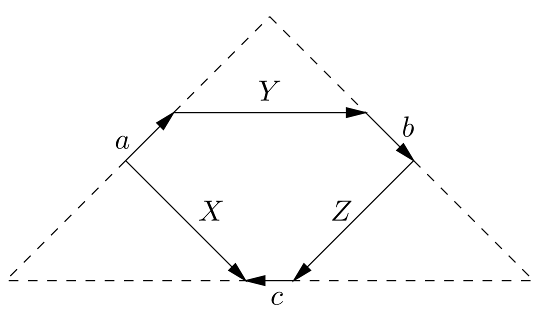

FIGURE 3. Another geodesic triangle.

We see that $\mathcal{G}$ generates a context–free language L which contains M and projects to 1 in G. It follows that $M = L \cap R\#R\#R$; and as intersections of context–free and regular languages are context–free, M is context–free.

5. Context–Free Implies Hyperbolic

In this section we complete the proof of Theorem 1. Let $\Sigma^* \to G$ be a choice of generators, R a regular combing, and M the multiplication table determined by R. Assume M is context–free. By Theorem 8 it suffices to show that for some combing R' every R'–triangle is δ–thin.

Let $\mathcal{G}$ be a context–free grammar for M in Chomsky normal form. This normal form condition means that the productions of $\mathcal{G}$ look like $A \to BC$ or $A \to a$ where $a \in \Sigma_\#$ and A, B, C are nonterminals. Without loss of generality we may assume that each production participates in some derivation of a word in M and that each nonterminal occurs in a production. For each nonterminal A let L_A be the context–free language of all terminal words derived from A by applying productions of $\mathcal{G}$. Our conditions guarantee that L_A is nonempty. Define u_A to be a shortest word in L_A, and let K be any constant greater than the length of every u_A.

We claim that for a fixed nonterminal A each word in L_A represents the same element of G. Indeed A occurs in a derivation of some $u\#v\#w \in M$ and derives a subword x of $u\#v\#w$. Because of the way derivations are defined for context–free grammars, replacing x by any $y \in L_A$ yields another member of M. As all elements of M represent 1 in G (recall that $\overline{\#} = 1$), it follows that $\overline{x} = \overline{y}$. The same reasoning shows that every word in L_A contains the same number of #'s. Define that number to be the rank of A.

Fix an R–triangle T with label $u\#v\#w \in M$; uvw is the label of a cycle $g_1, \dots, g_n$ in G, and subwords of uvw are paths. We may think of each letter in uvw as joining two group elements in the cycle.

Pick a letter b in uvw. We will estimate the distance from the group elements it joins to another side of T. Any derivation of $u\#v\#w$ can be written as $S \xrightarrow{*} \alpha A \beta \to \alpha BC\beta \xrightarrow{*} u\#v\#w$ where A is the last nonterminal of positive rank which derives a subword x_A containing b, and B or C is a nonterminal of rank zero deriving subword of x_A containing b. Assume B has rank zero and derives a subword (in fact a prefix) x_B of x_A containing b. The argument is the same in the other case. The situation is illustrated in Figure 4 for the case that b lies in v. The dashed lines in Figure 4 represent subwords of $u\#v\#w$.

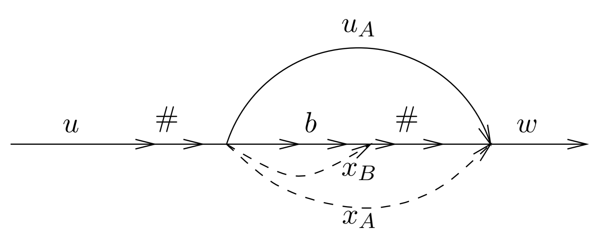

Figure 4. Estimating δ.

Since x_A contains one or two #'s, it begins on one side of T and ends on another. There is a path u_A with the same initial and terminal point as

x_A, and consequently the distance from b to another side of T is at most $|x_B| + |u_A| \leq |x_B| + K$. Thus the following lemma completes the proof of Theorem 1.

Lemma 10. *There exists a constant K' and a regular combing $R' \subset R$ such that every word in the restriction of M to R' affords a derivation in which each nonterminal of rank zero derives a subword of length at most K'.*

Proof. By the Pumping Lemma for context–free languages there exists a constant K' such that for all A each $z \in L_A$ of length $|z| \geq K'$ contains a subword x of length at most K' which can be replaced by a shorter word y to obtain another element of L_A. Note that $\overline{x} = \overline{y}$ as all words in L_A represent the same element of G.

By Lemma 4 $\rho_{x,y} = \{(pxq, pyq) \mid p, q \in \Sigma_{\#}^*\}$ is a rational transduction. Hence so is ρ, the union of $\rho_{x,y}$ over all pairs x, y which occur above for nonterminals of rank 0. Define $R' = R - \rho^{-1}(R)$. R' consists of all words in R which cannot be reduced to other words in R by a substitution of the form $x \to y$. Since these reductions are length reducing, R' contains all words in R which are of minimal length among words in R representing the same element of G. Consequently R' is still a combing. R' is regular as it is a difference of regular languages.

Let $M' = M \cap (R'\#R'\#R')$, and consider a derivation $S \xrightarrow{*} u\#v\#w \in M'$. Each nonterminal of rank zero appearing in this derivation derives a subword z of u,v, or w. If $|z| > K'$, then one of u, v, w can be shortened by a substitution of y for x contrary to our choice of $u\#v\#w$. $\square$

6. The Combing Σ^*

This section is devoted to the proof of Theorem 8. Consider a choice of generators $\Sigma^* \to G$. Let $W = \{w \mid \overline{w} = 1\}$ be the word problem and M be the multiplication table determined by the combing Σ^*. It is well known that G is finite if and only if W is regular (see [10] for example), and by [15] together with [5] G is virtually free if and only if W is context–free. Thus it is enough to show M is regular if and only W is and M is context-free if and only if W is. We give the argument for the regular case. The argument for the context–free case is exactly the same.

Let W_1 be the inverse image of 1 under the extended homomorphism $\Sigma_{\#}^* \to G$. Observe that $W_1 = f^{-1}(W)$ where $f : \Sigma_{\#}^*$ to Σ^* is defined by $f(\#) = \epsilon$ and $f(a) = a$, $a \in \Sigma$.

Suppose W is regular; then $M = W_1 \cap \Sigma^*\#\Sigma^*\#\Sigma^* = f^{-1}(W) \cap \Sigma^*\#\Sigma^*\#\Sigma^*$ is also regular. Conversely if M is regular, then $W = f(W\#\#) = f(M \cap \Sigma^*\#\#)$ is regular too.

7. Automatic Groups

In this section we prove Theorem 3. First suppose G admits an asynchronous automatic structure based on a combing R which is closed under taking

inverses. For each $a \in \Sigma_\epsilon$ the relation $\rho_a = \{(u,v) \mid u,v \in R, \overline{ua} = \overline{v}\}$ is a rational transduction. By Lemma 6 $L_a = \{u\#w \mid u, w^{-1} \in R, \overline{ua} = \overline{w}^{-1}\}$ is context–free. But as R is closed under inverses, $L_a = \{u\#w \mid u,w \in R, \overline{uaw} = 1\} = C(\overline{a})$.

To prove the converse fix $a \in \Sigma_\epsilon^*$, pick a context–free grammar $\mathcal{G}$ for $C(\overline{a})$ in Chomsky normal form, and argue as in Section 5. We may assume that $\mathcal{G}$ has no superfluous nonterminals or productions and consequently that the nonterminals of $\mathcal{G}$ each have a well defined rank of zero or one. By expanding nonterminals of rank one first we can put every derivation into the form $S \xrightarrow{*} \alpha B \beta \xrightarrow{*} u\#w$ where S and B have rank one, α and β are words in nonterminals of rank zero, $\alpha \xrightarrow{*} u$, $\beta \xrightarrow{*} w$, and $B \to \#$. Further for some regular subcombing $R_a \subset R$, $u\#w \in C(\overline{a}) \cap (R_a \times R_a)$ implies that subwords of u or w derived from nonterminals in α or β have length at most K.

Consider the linear context–free grammar $\mathcal{G}'$ obtained by replacing each production $A \to BC$ of $\mathcal{G}$ in which A and C are of rank one and B of rank zero by the productions $A \to xC$ where x ranges over all words of length at most K in L_B. Likewise productions $A \to BC$ with A and B of rank one and C of rank zero are replaced by productions $A \to By$, $y \in L_C$, $|y| \leq K$. Let L be the language generated by $\mathcal{G}'$. Clearly $L \subset C(\overline{a})$, and from the discussion above it follows that $C(\overline{a}) \cap (R_a \times R_a) \subset L$.

Note that nonterminals of rank zero do not appear in any $\mathcal{G}'$–derivations of words in L. After all nonterminals of rank zero are deleted $\mathcal{G}'$ satisfies the hypothesis of Lemma 6, and it follows that $\tau_a = \{(u,v) \mid u\#v^{-1} \in L\}$ is a rational transduction.

By construction each R_a contains all words of minimal length among those in R defining the same element of G. Thus $R_1 = \cap R_a$ is a regular combing, and so is $R' = R \cup R_1^{-1}$. Replace each τ_a by its restriction to $R_1 \times R_1^{-1}$; τ_a is still a regular transduction. Let $\mu = \tau_\epsilon^{-1}$, and check that $\tau_a \cup (\tau_a \circ \mu) \cup (\mu \circ \tau_a) \cup (\mu \circ \tau_a \circ \mu) = \{(u,v) \mid u,v \in R', \overline{ua} = \overline{v}\}$. It follows from Lemma 7 that the combing R' supports an asynchronous automatic structure for G.

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